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APPROXIMATION BY PIECEWISE POLYNOMIAL

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Abstract: There have approach to approximate a given function by means of piecewise polynomial functions. This is because polynomial of interpolation in general may not approximate a given function. This deficiency is tackled by means of piecewise polynomial functions knotted at the partition points with some conditions, mainly by continuity.

Keywords: Continuity, Convexity and Interpolation.

1. INTRODUCTION

We study approximation of functions C^2 by means of quartic piecewise polynomial. Here we lighten the continuity requirements at the knot points and impose a convex type of structure inside the partition. The convex type condition is the following:

$$\alpha s(x_{i-1} + m_1 h) + (1 - \alpha)s(x_{i-1} + m_2 h) = s(x_{i-1} + \ h(\alpha m_1 + (1 - \alpha)m_2)),$$

$i = 1, 2, ...n$, where α, m_1, and m_2 are positive numbers such that $0 < m_1 < m_2 < 1$ and $0 < \alpha < 1$. We denote such class of spline functions by $S(4^c_1, \Delta)$. $S(3^c_1, \Delta)$ has a similar meaning for cubic case, Kumar and Das [5]. This consideration gives a coefficient matrix for computation of spline function.

Let us consider two consecutive subinterval of the partition namely $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. If we define a polynomial $p_t(x)$ of degree t in the first interval and in the second interval the other polynomial is $p_t(x) + c$, $c \neq 0$. At the common point the value of the first polynomial is $p_t(x_i)$ and the second is $p_t(x_i) + c$. Such defined piecewise function is not continuous at the common point. But the first derivative of the piecewise functions are continuous at x_i. In

view of this we consider the polynomial of this type of the next theorem.

2. MAIN RESULT

THEOREM 1: We consider a pp function s, where s and s'' are continuous, satisfying the interpolatory conditions:

(i) $s(x_i) = f(x_i)$, $i = 1, 2, ...n$,

(ii) $s(t_i) = f(t_i)$, $i = 1, 2, ...n$,

where $t_i = x_{i-1} + \theta h$, $0 < \theta < 1$,

and

(iii) $\alpha s(x_{i-1} + m_1 h) + (1 - \alpha) s(x_{i-1} + m_2 h) = s(x_{i-1} + \ h(\alpha m_1 + (1 - \alpha) m_2))$, $i = 1, 2, ...n$,

for uniform partition with $m_1 + m_2 = 1$, $0 < \alpha < 1$.

This pp function exists uniquely for $\theta = \dfrac{1}{4}$, if $\lambda_1 < m_2 < \lambda_2$ and

$\lambda_2 < m_2 < \lambda_1$,

where

$$\lambda_1 = \frac{1}{2\left(-80\alpha + 80\alpha^2 + 40\right)} \left(21 + 80\alpha^2 - 42\alpha + \sqrt{(601 + 1924\alpha^2 - 1924\alpha)}\right),$$

$\alpha < \dfrac{3}{4}$,

and

$$\lambda_2 = \frac{1}{2\left(-80\alpha + 80\alpha^2 + 40\right)}\left(21 + 80\alpha^2 - 42\alpha - \sqrt{(601 + 1924\alpha^2 - 1924\alpha)}\right),$$
$\alpha > \dfrac{3}{4}$.

THEOREM 2: The error of approximation of the pp function of Theorem 1 for $f \in C^2[0,1]$, is the following:

$$e(x) \leq \; h^2(\text{constant } \omega(f^{"};h) + \| D^{-1} \| \| f^{"} \|),$$

where constant $= \| D^{-1} \| \, (| \, p \, | + | \, r \, | + \theta + 2)$.

The estimates for $\| D^{-1} \|$ have been obtained at $\lambda_1 < m_2 < \lambda_2$,

$\| D^{-1} \| \leq \dfrac{12A}{T_4}$ and at $\lambda_2 < m_2 < \lambda_1$, $\| D^{-1} \| \leq \dfrac{12A}{T_5}$.

where

$$T_4 = \left(\frac{165}{64} - \frac{165}{32}\alpha + \frac{165}{32}\alpha^2\right) m_2^2 + \left(\frac{387}{64}\alpha - \frac{165}{32}\alpha^2 - \frac{387}{128}\right) m_2$$
$$-\frac{51}{128} - \frac{111}{64}\alpha + \frac{165}{128}\alpha^2$$

and

$$T_5 = \left(\frac{105}{64} - \frac{105}{32}\alpha + \frac{105}{32}\alpha^2\right) m_2^2 + \left(\frac{81}{16}\alpha - \frac{105}{32}\alpha^2 - \frac{81}{32}\right) m_2$$
$$-\frac{3}{8} - \frac{219}{128}\alpha + \frac{105}{128}\alpha^2.$$

Remarks:

1. In view of the complicated equations in the proof we considered $\theta = \dfrac{1}{4}$. The other values of θ can be considered.

2. We observe after the proof of Theorem 2 that the order of the $e_i^{"}$ cannot be improved in general.

Proof of the Theorem 1:

Let $s_i(x) = s(x)$ be pp function in $[x_{i-1}, x_i]$, $i = 1, 2, ...n$. We write $s^{"}(x_i) = M_i$, $i = 1, 2, ...n$.

We have

$$\textbf{(1)}\ s^{"}(x)h^2 = -M_{i-1}(2(x_i - x)(x - x_{i-1}) - (x_i - x)^2)$$
$$-M_i(2(x_i - x)(x - x_{i-1}) - (x - x_{i-1})^2) + 6\eta_i(x_i - x)(x - x_{i-1}),$$

where $\eta_i^{\prime}s$ are constants.

The pp function $s(x)$ defined on interval $[x_{i-1}, x_i]$ is given by

$$\textbf{(2)}\quad s(x)h^2 = -M_{i-1}(\frac{1}{3}(x_i - x)(x - x_{i-1}))^3 + \frac{1}{6}(x - x_{i-1})^4$$
$$-\frac{1}{12}(x_i - x)^4) - M_i(\frac{1}{3}(x_i - x)(x - x_{i-1})^3 + \frac{1}{12}(x - x_{i-1})^4)$$
$$+\eta_i((x_i - x)(x - x_{i-1})^3 + \frac{1}{2}(x - x_{i-1})^4) + \delta_i(x - x_{i-1})h^3 + \gamma_i h^4),$$

where $\delta_i^{\prime}s$ and $\gamma_i^{\prime}s$ are also constants.

Now by using interpolatory condition (i) of $s(x)$, we get

$$\textbf{(3)}\quad \gamma_{i+1} - \gamma_i = h^{-2}(f(x_i) - f(x_{i-1})) - \frac{1}{12}(M_i - M_{i-1}).$$

Now apply the other interpolatory condition (ii) of $s(x)$. We find

$$\textbf{(4)}\quad \delta_{i+1} - \delta_i = \theta^{-1}(h^{-2}(f(t_{i+1})\ - f(t_i)) + M_{i+1}b + M_i(a - b)$$
$$-M_{i-1}a - (\eta_{i+1} - \eta_i)c - (\gamma_{i+1} - \gamma_i)).$$

where

$$a = \frac{1}{3}(1 - \theta)\theta^3 + \frac{1}{6}\theta^4 + \frac{1}{12}(1 - \theta)^4\ ,$$
$$b = \frac{1}{3}(1 - \theta)\theta^3 + \frac{1}{12}\theta^4\ ,$$
$$c = (1 - \theta)\theta^3 + \frac{1}{2}\theta^4.$$

We get the restriction about the continuity of $s(x)$, we find

(5) $\eta_{i+1} - \eta_i = -\frac{B}{A}M_{i-1} + (\frac{B-C}{A})M_i + \frac{C}{A}M_{i+1},\ i = 1, 2, ..., n-1.$

where

$$A = \frac{1}{2}(\alpha^2 - 4\alpha^2 m_2 + 4\alpha^2 m_2^2 - 4\alpha m_2^2 + 4\alpha m_2 - \alpha + 2m_2^2 - 2m_2 - 1),$$
$$B = \frac{1}{12}(12\alpha^2 m_2^2 + 3\alpha^2 - 12\alpha^2 m_2 - 12\alpha m_2^2 + 16\alpha m_2 - 5\alpha + 6m_2^2 - 8m_2 + 1),$$

and

$$C = \frac{1}{12}(-12\alpha^2 m_2 + 12\alpha^2 m_2^2 + 3\alpha^2 - 12\alpha m_2^2 + 8\alpha m_2 - \alpha + 6m_2^2 - 4m_2 - 1)\ .$$

From equations (3),(4) and (5), we obtain the following system of equations, for $i = 1, 2, ...n$:

(6) $pM_{i+1} \ + qM_i + rM_{i-1}$

$$= h^{-2}\theta\{f(x_{i+1}) - f(x_i) - \theta^{-1}(f(t_{i+1}) - f(t_i)) - (1-\theta^{-1})(f(x_i) - f(x_{i-1}))\},$$

for $i = 1, 2, .., n$, where

$$p = \frac{1}{12A}\ (-A\theta + 12Ab + 6C\theta - 12Cc),$$
$$q = \frac{1}{12A}(-2A\theta + 12Aa - 12Ab + 6B\theta - 6C\theta - 12Bc + 12Cc + A),$$
$$r = \frac{1}{12A}(3A\theta - 12Aa - 6B\theta + 12cB - A).$$

The coefficients of $M_i's$ of system of equations becomes complicated for general θ. We consider a special case at θ, i.e. $\theta = \frac{1}{4}$.

We observe

(7) $A < 0$ for all α and m_2.

The following is the coefficient matrix of $M_i's$ of the above equations (6),

$$D = C(p,q,r:p,r) = \begin{bmatrix} q & r & 0 & 0 & 0 & ... & 0 & p \\ p & q & r & 0 & 0 & ... & 0 & 0 \\ 0 & p & q & r & 0 & ... & 0 & 0 \\ 0 & 0 & p & q & r & ... & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ r & 0 & 0 & 0 & 0 & ... & p & q \end{bmatrix}.$$

We investigate non-singularity of the coefficient matrix D.

In order to see non singularity of the matrix D, we see the diagonal dominance of matrix D. The matrix D is diagonally dominant if

(8) $dd = q - r - \mid p \mid > 0,$

since the coefficients of M_i and M_{i-1} are positive for all permissible values of α and m_2.

For $\lambda_1 < m_2 < \lambda_2$, (8) can be written as

$q - r + p = \frac{1}{12A} T_4 > 0,$

It can be seen that always $T_4 < 0$. Since $A < 0$, we get that the matrix is diagonally dominant . Also $\parallel D^{-1} \parallel \leq \frac{1}{q-r+p} = \frac{12A}{T_4}$.

Now we consider the case $\lambda_2 < m_2 < \lambda_1$, (8) can be written as

$q - r - p \ = \frac{1}{12A} T_5,$

we see that

T_5 does not change sign in permissible values of m_2 and $T_5 < 0$. Hence the matrix is diagonally dominant and $\parallel D^{-1} \parallel \leq \frac{1}{q-r-p} = \frac{12A}{T_5}$.

Proof of Theorem 2:

Let $e(x) = s(x) - f(x)$.

Now by equation (6), we get

(9) $pe^{"}_{i+1} + qe^{"}_{i} + re^{"}_{i-1} = U_i - pf^{"}_{i+1} - qf^{"}_{i} - rf^{"}_{i-1}$, $i = 1, 2, ..., n$,

where

U_i

$= h^{-2}\theta\{(f(x_{i+1})_{-}f(x_i) - \theta^{-1}(f(t_{i+1}) - f(t_i)) - (1-\theta^{-1})(f(x_i) - f(x_{i-1}))\}$.

Equation (9) can be written as

$$
\begin{aligned}
pe^{"}_{i+1} + qe^{"}_{i+1} + re^{"}_{i+1}
&= \theta f^{"}(\delta_i) - f^{"}(\phi_i) - pf^{"}_{i+1} + (p+r)f^{"}_{i} - rf^{"}_{i-1},\\
&= \theta f^{"}(\delta_i) - f^{"}(\phi_i) - p(f^{"}_{i+1} - f^{"}_{i}) + r(f^{"}_{i} - f^{"}_{i-1}),\\
&= \theta(f^{"}(\delta_i) - f^{"}(\phi_i)) - (1-\theta)f^{"}(\phi_i) - p(f^{"}_{i+1} - f^{"}_{i}) + r(f^{"}_{i} - f^{"}_{i-1}),\\
&\leq (\mid p \mid + \mid r \mid + \theta)\omega(f^{"}; h) + \parallel f^{"} \parallel .
\end{aligned}
$$

We find

$$
\begin{aligned}
&pe^{"}_{i+1} + qe^{"}_{i} + re^{"}_{i-1}\\
&= \theta \exp(x_{i-1})(1 + \frac{h}{2} + \text{higher powers of } (h))(\frac{h}{2} + \text{ higher powers of } (h))\\
&-(\frac{\theta h}{2} + \text{ higher powers of } (\theta h)) + o(1).
\end{aligned}
$$

We have

$$
\begin{aligned}
&min\ \{\mid pe^{"}_{i+1} \mid, \mid qe^{"}_{i} \mid, \mid re^{"}_{i-1} \mid\}\\
&\geq\ \theta \exp(x_{i-1})(1 + \frac{h}{2} + \text{ higher powers of } (h))(\frac{h}{2} + \text{ higher powers of } (h))
\end{aligned}
$$

$$-(\frac{\theta h}{2} + \text{higher powers of } (\theta h)) + o(1)$$

$$\geq \ \theta \exp(x_{i-1}) + o(1).$$

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