LA – SEMIRINGS In Which (S,.) Is Anti-Inverse Semigroup
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Abstract:
This paper deals with the some results on LA– Semirings in which(s, •) is anti-inverse semigroup. In the first case of the LA - Semiring (S, +, .) satisfying the identity a+1 = 1, for all a in S then it is proved that (S, +) is anti-inverse semigroup. It is also proved that if (S, +,•) is a LA – Semiring satisfying the above identity then (S, +) is an abelian semigroup and sum of two anti inverse elements is again anti inverse element in (S, +) and also proved that (S, +, •) is medial semiring. In this second case we consider LA- semiring (S, +, •) in which (s, •) is anti-inverse semigroup satisfying the identity a+1= a for all a in S then (S, +) is anti inverse semigroup and sum of two inverse elements is again anti inverse element in (S, +). It is also proved that in LA semiring in which (s, •) is anti-inverse semigroup then (S, +) is an abelian semigroup and the product of two inverse elements is again inverse element in (s, •).

Keywords — LA-Semigroup, LA-semirings, Anti-inverse semigroup

Introduction:
LA- semirings are naturally developed by the concept of LA- semigroup. The concepts of LA – semigroup was introduced by M.A. Kazim and M. Naseeruddin [1] in 1972. Since then lot of papers has been presented on LA - semigroups like. Mushtaq, Q and Khan [02], M Mustaq, Q. and yousuf, S.M. [03], Qaiser Mushtaq[04]. Anti inverse semigroups are studied by S-Bogdanovic S.Milic V.Pavloric. In this paper mainly we concentrate on the structures of anti inverse semigroup in LA-semirings. We determine some structures of LA-semirings in which the multiplicative structure is anti inverse semigroup.

1.1.1. Definition: A left almost semigroup (LA-semigroup) or Abel-Grassmanns groupoid (AG-groupoid) is a groupoid S with left invertive law: (ab)c = (cb)a for all a,b,c ∈ S

Example:- Let S = {a, b, c} the following multiplication table shows that S is a LA-Semigroup.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

1.1.2. Definition: A semiring (S, +, •) is said to be LA-Semiring if
1. (S, +) is a LA-Semigroup
2. (S, •) is a LA-Semigroup

Example: Let S = {a, b, c} is a mono semiring with the following tables 1, 2 which is a LA-Semiring
1.2.1. Definition: A semi group S is called anti inverse if every element of S is anti inverse element.

Example: Let S = {a, b} then \((S, \cdot)\) with following table 1 or 2 or 3 is an inverse semigroup

\[
\begin{array}{ccc}
  + & a & b & c \\
  a & a & a & a \\
  b & a & a & c \\
  c & a & a & a \\
\end{array}
\]

\[
\begin{array}{ccc}
  \cdot & a & b & c \\
  a & a & a & a \\
  b & a & a & c \\
  c & a & a & a \\
\end{array}
\]

\[= x + x + axa + a = x(1+1) + axa + a = x + (ax+1)a = x + a = axa + a = (ax + 1)a = 1.a = a \]

\[\therefore \ x + a + x = a \]

Similarly \(a + x + a = x\)

\[\therefore (S, +) \text{ is a anti-inverse semigroup} \]

1.2.2. Theorem: Let \((S, +, \cdot)\) be a LA-semiring in which \((S, \cdot)\) is anti-inverse semi group and satisfying the identity \(a+1=1 \ \forall a \in S\) then \((S, +)\) is anti-inverse semigroup.

Proof: Let \((S, +, \cdot)\) be a LA-Semiring and \((S, \cdot)\) be anti-inverse semigroup satisfying the identity \(a+1=1 \ \forall a \in S\)

Let \(a \in S\), since \((S, \cdot)\) is anti-inverse there exist an element \(x \in S\) such that \(xa = a\).

Consider \(x+a+x = x+a.1+x\)

\[= x + a(1 + xa) + x \]

\[= x + (a + axa + x) \]

\[= x + x + axa + a = x(1+1) + axa + a = x + (ax+1)a = x + a = axa + a = (ax + 1)a = 1.a = a \]

\[\therefore \ x + a + x = a \]

Similarly \(a + x + a = x\)

\[\therefore (S, +) \text{ is a anti-inverse semigroup} \]

1.2.3. Theorem : Let \((S, +, \cdot)\) be a LA-semiring and \((S, \cdot)\) be anti-inverse semi group then the product of two anti-inverse elements is also anti-inverse element in \((S, \cdot)\)

Proof: Let \((S, +, \cdot)\) be a LA-semiring and \((S, \cdot)\) be an anti-inverse semi group.

Let \(a, b\) are two elements in \((S, \cdot)\) then there exist \(x, y\) in \(S\) such that \(xax = a, yby = b\)

Consider \(yx \ ab \ yx = byb \ xabyaxa \)

\[= byb \ xab \ yaxa = by(bxa) \ byaxa \]

\[= by(ax(b \ by \ a)xa = by(ax \ a) \ (by \ b)xa = by \ xy \ xa \]

\[\therefore (S, \cdot) \text{ is a anti-inverse semigroup} \]
\[= byx (yxa)\]
\[= by(xax)y\]
\[= (bya)y\]
\[= ayby\]
\[= ab\]

\[\therefore \ yx \ ab \ yx = ab\]

Similarly we can prove that \(ba \ xy \ ba = xy\)

Hence the product of two anti-inverse elements is again anti-inverse element in \((S, \cdot)\)

**1.2.4. Theorem:** Let \((S, +, \cdot)\) be a LA-Semiring and \((S, \cdot)\) be anti-inverse semigroup then \((S, \cdot)\) is an abelian semigroup.

**Proof:** Let \((S, +, \cdot)\) be a LA-semiring and \((S, \cdot)\) is an anti-inverse semigroup.

From the above theorem for any \(a, b \in S\), there exist \(x, y \in S\) such that

\[yx \ a \ byx = ab\]
\[y \ xa \ (byx) = ab\]
\[y \ xa \ xyb = ab\]
\[y(ayb) = ab\]
\[y \ b \ y \ a = ab\]
\[ba = ab\]

Hence \((S, \cdot)\) is an abelian semigroup.

**1.2.5. Theorem:** Let \((S, +, \cdot)\) be a LA-semiring in which \((S, \cdot)\) is an anti-inverse semigroup and satisfying the identity \(a+1= 1 \ \forall a \in S\). Then by the theorem 3.2.2 \((S, +)\) is anti-inverse semigroup.

Let \(a, b \in S\) then there exists \(x, y \in S\) such that \(x+a+x = a, y+b+y = b\) and \(a+x+a = x, b+y+b = y\)

Consider
\[y+x+a+b+y+x = b+y+b+x+a+b+y+a+x+a\]
\[= b+y+(b+x+a)+b+y+a+x+a\]
\[= b+y+a+x+(b+b+y)+a+x+a\]
\[= b+y+(a+x+a)+(b+y+b)+x+a\]
\[= b+y+(x+a+x)+y\]
\[= (b+y+a)+y\]
\[= a+y+b+y\]
\[= a+b\]
\[\therefore \ y+x+a+b+y+x=a+b \quad \ldots(1)\]

Similarly we can prove that
\[b+a+x+y+b+a = x+y\]
\[\therefore \ a+b \text{ is an anti-inverse element in } (S, +)\]

Therefore the sum of two anti-inverse elements is again anti-inverse element in \((S, +)\).

To show that \((S, +)\) is an abelian semigroup.

From equation (1) \[a+b = y+x+a+(b+y+x)\]
\[= y+(x+a+x)+y+b\]
\[= y+(a+y+b)\]
\[= (y+b+y)+a\]
\[= b+a\]
\[\therefore \ a+b = b+a\]

Hence \((S, +)\) is an abelian semigroup

**1.2.7. Theorem:** Let \((S, +, \cdot)\) be a LA-Semiring which satisfies the identity \(a+1 = 1 \ \forall a \in S\). If \((S, \cdot)\) is an anti-inverse semigroup then \((S, +, \cdot)\) is a medial semiring.
**Proof:** Let \((S, +, \cdot)\) be a LA-Semiring satisfying the identity \(a+1 = 1, \forall a \in S\). Let \((S, \cdot)\) be an anti-inverse semigroup with

From the theorems 3.2.4 and 3.2.5 we have \((S, \cdot)\) and \((S, +)\) are abelian semigroups.

Let \(a, b, c, d \in (S, \cdot)\) then

\[
abcd = a(bc)d = a(cb)d
\]

\[
abcd = a\ cb\ d
\]

\[\therefore\ \text{(S, \cdot) is a medial semigroup}\]

Similarly \((S, +)\) is also a medial semigroup

Hence \((S, +, \cdot)\) is a medial semiring

**1.2.8. Theorem:** Let \((S, +, \cdot)\) be a LA-semiring in which \((S, \cdot)\) is an anti-inverse semigroup and satisfying the identity \(a+1\) = a, \(\forall a \in S\) then \((S, +)\) is an anti-inverse semigroup.

**Proof:** Let \((S, +, \cdot)\) be a LA-semiring satisfying the identity \(a+1 = a, \forall a \in S\). Let \((S, \cdot)\) be an anti-inverse semigroup.

Since \((S, \cdot)\) is an anti-inverse semigroup then for any \(a \in S\) therexist \(x \in S\) such that \(xax = a\) and \(axa = x\)

Consider \(x+a+x = x+a+x\)

\[
= x+xax+x
\]

\[
= x(1+ax)+x
\]

\[
= xax+x
\]

\[
= (xa+1)x
\]

\[
= xa.x = xax = a
\]

\[
x+a+x = a
\]

Similarly we can prove that \(a+x+a = x\)

Hence \(a\) is an anti-inverse element is \((S, +)\)

Therefore \((S, +)\) is an anti-inverse semigroup.

**1.2.9. Theorem:** Let \((S, +, \cdot)\) be a LA-semiring in which \((S, \cdot)\) is an anti-inverse semigroup and satisfying the identity \(a+1\) = a, \(\forall a \in S\) then the sum of two anti-inverse elements is also anti inverse element in \((S, +)\).

**Proof:** Proof is similar to theorem 3.2.5

**Reference:**


[04] Qaiser Mushtaq “left almost semi groups defined by a free algebra”. and Muhammad Inam Quasi groups and related systems 16 (2008) 69-76

[05] Sreenivasulu Reddy,P. & Shobhalatha,G. Some studies on Regular semigroups (Thesis)