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On the non-homogeneous ternary cubic equation $3(x^2+y^2)-5xy+x+y+1=111z^3$

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Abstract:

The non-homogeneous cubic equation with three unknowns represented by $3(x^2+y^2)-5xy+x+y+1=111z^3$ is analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

Keywords — **Non-homogeneous cubic, ternary cubic, integer solutions.**

I. INTRODUCTION

It is well known that the Diophantine equations are rich in variety [1-3]. In particular, one may refer [4-11] for cubic with three unknowns. In this paper, yet another cubic equation with three unknowns given by $3(x^2+y^2)-5xy+x+y+1=111z^3$ is considered for determining its infinitely many non-zero integer solutions. Also, A few interesting relations among the solutions are exhibited.

II. NOTATIONS

- $SO_n = n(2n^2-1)$ - Stella octangular number of rank n
- $CP_{6,n} = n^3$ - Centered hexagonal pyramidal number of rank n
- $GNO_n = 2n-1$ - Gnomonic number of rank n
- $PR_n = n(n+1)$ - Pronic number of rank n
- $t_{m,n} = n\left[1+\frac{(n-1)(m-2)}{2}\right]$ - Polygonal number of rank n with size m
- $S_n = 6n^2-6n+1$ - Star number of rank n
- $R_n = 4n^3-6n^2+4n-1$ -Rombic Dodecagonal number of rank n

III. METHOD OF ANALYSIS

The ternary non-homogeneous cubic equation to be solved is

$$3(x^2+y^2)-5xy+x+y+1=111z^3 \tag{1}$$

Introducing the linear transformations

$$x=u+v,\quad y=u-v \tag{2}$$

in (1), it gives

$$U^2+11v^2=111z^3 \tag{3}$$

where $U=u+1$ (4)

Assume $z=z(a,b)=a^2+11b^2$ (5)

Solving (3) through various methods and using (2), different sets of integer solutions to (1) are obtained.

A.Method 1:

Consider

$$111=(10+i\sqrt{11})(10-i\sqrt{11}) \tag{6}$$

Substituting (5), (6) in (4) and applying the method of factorization,

$$(U+i\sqrt{11}v)(U-i\sqrt{11}v)=(10+i\sqrt{11})(10-i\sqrt{11})*(a+i\sqrt{11}b)^3(a-i\sqrt{11}b)^3$$

Equating the positive and negative terms in the above equation, we have

$$(U+i\sqrt{11}v)=(10+i\sqrt{11})(a+i\sqrt{11}b)^3 \tag{7}$$

$$\left(U - i\sqrt{11}v\right) = \left(10 - i\sqrt{11}\right)\left(a - i\sqrt{11}b\right)^3 \quad (8)$$

Equating the real and imaginary parts in either (7) or (8), we have

$$U = 10a^3 - 330ab^2 - 33a^2b + 121b^3 \quad (9)$$

$$v = a^3 - 33ab^2 + 30a^2b - 110b^3 \quad (10)$$

Substitution of (9) in (4) gives

$$u = 10a^3 - 330ab^2 - 33a^2b + 121b^3 - 1 \quad (11)$$

Substituting the above values of u and v in (2), we get

$$\left.\begin{aligned} x = x(a,b) = 11a^3 - 363ab^2 - 3a^2b + 11b^3 - 1 \\ y = y(a,b) = 9a^3 - 297ab^2 - 63a^2b + 231b^3 - 1 \end{aligned}\right\} \quad (12)$$

Thus, (5) and (12) represents the integer solutions of (1).

Properties:

- $x(1,b) - 11CP_{b,6} + t_{728,b} + 365b - 10 = 0$
- $y(a,a) + 120CP_{a,6} + 1 = 0$
- $\left.\begin{aligned} x(a,a+1) + R_a + 340CP_{a,6} + 702PR_a \\ -188GNO_a - 197 = 0 \end{aligned}\right\}$
- $x(a,1) - y(a,1) - SO_a - 60PR_a + 125a + 220 = 0$
- $y(a,-a) + 456CP_{a,6} + 1 = 0$

Note 1:

Apart from (6), 111 is also expressed as

$$111 = \frac{\left(13 + i5\sqrt{11}\right)\left(13 - i5\sqrt{11}\right)}{4} \quad (13)$$

In this case, the corresponding solutions to (1) are given by

$$x = x(A,B) = 72A^3 - 504A^2B - 2376AB^2 + 1848B^3 - 1$$

$$y = y(A,B) = 32A^3 - 816A^2B - 1056AB^2 + 2992B^3 - 1$$

$$z = z(A,B) = 4A^2 + 44B^2$$

B.Method 2:

(3) is written as

$$U^2 + 11v^2 = 111z^3 * 1 \quad (14)$$

Assume $1 = \dfrac{\left(5 + i\sqrt{11}\right)\left(5 - i\sqrt{11}\right)}{36}$ (15)

Substituting (5), (6) and (15) in (14) and employing the method of factorization, define

$$\left(U + i\sqrt{11}v\right) = \frac{1}{6}\left(10 + i\sqrt{11}\right)\left(5 + i\sqrt{11}\right)\left(a + i\sqrt{11}b\right) \quad (16)$$

Equating the real and imaginary parts in (16), we have

$$\left.\begin{aligned} U = \frac{1}{6}\left(39a^3 - 1287ab^2 - 495a^2b + 1815b^3\right) \\ v = \frac{1}{6}\left(15a^3 - 495ab^2 + 117a^2b - 429b^3\right) \end{aligned}\right\} \quad (17)$$

Replacing a by $6A$ and b by $6B$ in (17) and (5) we get

$$U = 1404A^3 - 46332AB^2 - 17820A^2B + 65340B^3 \quad (18)$$

$$v = 540A^3 - 17820AB^2 + 4212A^2B - 15444B^3 \quad (19)$$

$$z = 36A^2 + 396B^2 \quad (20)$$

Substitution of (18) in (4) gives

$$\left.\begin{aligned} u = 1404A^3 - 46332AB^2 - 17820A^2B \\ + 65340B^3 - 1 \end{aligned}\right\} \quad (21)$$

Substituting the above values of u and v in (2), it is seen that

$$\left.\begin{aligned} x = x(A,B) = 1944A^3 - 64152AB^2 - 13608A^2B \\ + 49896B^3 - 1 \\ y = y(A,B) = 864A^3 - 28512AB^2 - 22032A^2B \\ + 80784B^3 - 1 \end{aligned}\right\} \quad (22)$$

Thus, (20) and (22) represents the integer solutions of (1).

Properties:

- $x(A,A) + 25920CP_{A,6} + 1 = 0$
- $\left.\begin{aligned} y(B+1,B) - 106272CP_{B,3} + 47952PR_B \\ + 23328GNO_B + 44497 = 0 \end{aligned}\right\}$
- $x(1,B) - 24948SO_B + t_{128306,B} + 52811B - 1943 = 0$
- $\left.\begin{aligned} x(1,B) - y(1,B) + 30888CP_{B,6} + 35640PR_B \\ - 22032(GNO_B) - 23112 = 0 \end{aligned}\right\}$
- $\left.\begin{aligned} y(A,1) - 864CP_{A,6} - S_A + 22038PR_A \\ + 3234GO_A - 77548 = 0 \end{aligned}\right\}$

Note 2:

It is to be noted that, in addition to (15), 1 may also be represented as

$$1 = \frac{\left(1 + i3\sqrt{11}\right)\left(1 - i3\sqrt{11}\right)}{100} \quad (23)$$

For this choice, the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 800A^3 - 26400AB^2 - 109200A^2B \\ &\quad + 400400B^3 - 1 \\ y = y(A,B) &= -5400A^3 + 178200AB^2 - 95400A^2B \\ &\quad + 349800B^3 - 1 \\ z = z(A,B) &= 100A^2 + 1100B^2 \end{aligned}\right\}$$

Note 3:

In (14), employing (13) along with (15) and (23) inturn, one obtains two more sets of integer solutions to (1) which are exhibited below:

Set 1:

$$\left.\begin{aligned} x = x(A,B) &= 6912A^3 - 228096AB^2 - 176256A^2B \\ &\quad + 646272B^3 - 1 \\ y = y(A,B) &= -4032A^3 + 133056AB^2 - 184896A^2B \\ &\quad + 677952B^3 - 1 \\ z = z(A,B) &= 144A^2 + 1584B^2 \end{aligned}\right\}$$

Set 2:

$$\left.\begin{aligned} x = x(A,B) &= -43200A^3 + 1425600AB^2 - 763200A^2B \\ &\quad + 2798400B^3 - 1 \\ y = y(A,B) &= -78400A^3 + 2587200AB^2 - 398400A^2B \\ &\quad + 1460800B^3 - 1 \\ z &= 400A^2 + 4400B^2 \end{aligned}\right\}$$

In this paper, a search is performed to obtain different sets of integer solutions to the ternary cubic equation given by (1). To conclude, one may search for other choices of integer solutions to (1).

REFERENCES

[1] Dickson. L.E., "History of the Theory of Numbers", Vol 2, Diophantine analysis, New York, Dover, 2005.

[2] Mordell. L.J., "Diophantine Equations, Academic Press, New York, 1969.

[3] Carmichael. R.D., "The Theory of numbers and Diophantine Analysis" New York, 1959.

[4] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, Latice points on the Non- Homogeneous cubic equation $x^3 + y^3 + z^3 + (x+y+z) = 0$*, impact J.sci.Tech, 7(1), 21-25, 2013.*

[5] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, Latice points on the Non- Homogeneous cubic equation $x^3 + y^3 + z^3 - (x+y+z) = 0$*, impact J.sci.Tech, 7(1), 51-55,2013.*

[6] K.Meena, M.A.Gopalan, S.Vidhyalakshmi and S.Aarthy Thangam, On the ternary non-homogeneous cubic equation $4(x+y)^2 - 7xy + (x+y) + 15(x-y) = 16(z^3 - 1)$*, Bessel J Math.,4(3), 75-80, 2014.*

[7]] S.Vidhyalakshmi, M.A.Gopalan and A.Kavitha , On the ternary cubic equation $5(X+Y)^2 - 7XY + X + Y + 1 = 23Z^3$*, IJIRR, 1(10) , 99-101,2014.*

[8] M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, On the non-homogeneous ternary cubic Equation $2a^2(x^2 + y^2) - 2a(k+1)(x+y) + (k+1)^2 = 2^{2n} z^3$*, Universe of Emerging Technologies and science, 2(1), 1-5,2015,.*

[9] M.A.Gopalan, N.Thiruniraiselvi, A.Kavitha, On the ternary cubic Diophantine equation $7x^2 - 4y^2 = 3z^3$*, International Journal of Recent Scientific Research , 6(9) , 6197-61992015 .*

[10]M.A.Gopalan, N.Thiruniraiselvi and Sridevi, On The Ternary Cubic Equation $5(x^2 + y^2) - 8xy = 74(k^2 + s^2)z^3$*, IJMRME, 1(1), 317-323 2015.*

[11]M.A.Gopalan, Sharadha Kumar, On The Non-Homogeneous Ternary Cubic Equation $(x+y)^2 - 3xy = 12z^3$*, IJCESR, 5(1), 68-702018.*