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GENERALIZED STUDY OF UNIQUE FIXED POINTS AND COMMUTING SELFMAPS

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Abstract:

Some common fixed point theorems related to complete and compact metric space are proved in the present paper. Which includes the fixed point theorems given by B. E. Rhoades [31] , G. Jungck [21], G. Das et al. [9], and U. P. Dolhare [12, 13, 14, 15]. Fixed points for selfmappings, in complete metric spaces, are obtained by using weakly commuting pairs and commuting mappings. **W**e have extended and generalized the results obtained by K. Goebel and W. A. Kirk and proved fixed point theorems for selfmaps. Fixed point theorems for selfmappings satisfying weakly contractive conditions, involving altering distance, in complete metric space are also studied.

***Keywords* — Weakly commuting selfmaps, Complete metric space, Fixed points, Fixed point theorem.**

1 INTRODUCTION

Study of fixed points is the centre of research activity in Mathematics. And most of these results involve commuting selfmappings. One of the most important results about fixed points is given by Banach [2]. In 1930, Caccioppoli [5] generalized Banach contraction principle for the constant C_n . Edelstien [16], in 1962, studied periodic points and fixed points for contractive mappings. L. B. Ciric [7] introduced the notion of generalized contraction and proved some subsequent theorems. Browder [4], in 1976, studied nonlinear operators and nonlinear equations in Banach spaces. The concept of commuting selfmapping is useful for generalizing fixed point results. Generalization of results is a major research activity in fixed point theory and applications. The concept of commuting maps is developed in 1976 by G. Jungck [21]. G. Das and J. P, Dabata [9] applied it further to contractive type mappings. A. Meir and E. Keeler [26], in 1969, obtained a very important generalization of the Banach contraction principle. In 1986, G. Jungck [22] studied the common fixed points for compatible mappings. Shin-sen Chang [33] generalized some theorems for commuting selfmapping in complete metric spaces. R. P. Pant [29], in 1986, proved that, the common fixed point theorem of two pairs of commuting mappings satisfies Meir and Keeler type condition. M. A. Alghamdi et al. [1] generalized some fixed point theorems for commuting mappings. In the present paper we have used common fixed point theorems in commuting mappings for proving common fixed point results in complete metric space and also proved some interesting results on commuting selfmappings.

From many years, the study of fixed points of selfmappings satisfying contractive conditions fascinated many researchers. So it is one of the most active area of research activity. As a result of this, many fixed point results for selfmappings satisfying various types of contractive inequalities are developed by several researchers and authors some of them are in [2, 3, 4, 11, 12, 13, 21, 24, 25, 31]. B. E. Rhoades [31], in 1977, proved fixed point results for extended forms of contraction pairs. The notion of compatible mappings is introduced in 1986 by Jungck [22]. Rhoades and Jungck [23], in 1998, introduced the concept of weakly compatible maps. Fixed point results involving altering distances have been studied in [14]. V. I. Istratescu [19] in 1981, M. S. Khan et al. [25] in 1984, and K. Goebel et al. [18] in 1990 proved fixed point results for altering distances. J. Meszaros [27], K. P. R. Sastry and G. V. R. Babu [32] proved fixed point theorems in Metric spaces by altering distances between the points. S. V. R. Naidu [28] in 2003 and Singh et al. [34] in 2011 proved fixed point theorems for functions with altering distance. Popescu, in 2011, proved fixed point theorems involving weakly contractive type inequalities and weak contraction. In 1974 Lj. B. Ciric [8], and U. C. Gairola and Ram Krishan [17] in 2013 proved some fixed point results for self-maps satisfying a generalized weak contraction conditions. Many interesting results [10, 11, 30, 34] are obtained by various researchers and authors for the study of common fixed points of mappings satisfying some contractive type conditions.

In the present paper we introduce the generalized altering distance function and prove fixed point theorems for obtaining unique fixed point.

2 SOME CONCEPTS AND DEFINITIONS

Boyd and Wong, in 1969, defined commuting selfmaps as below.

Definition 2.1 : [3] Let (X , ϱ) be a metric space and f, g be self maps on X. Then f, g are said to be commuting if $fgx = gfx$, for all $x \in X$.

Jungck defined compatible selfmaps as follows.

Definition 2.2 : [21] Let f, g be mappings from a metric space (X , ϱ) into itself. Then f, g are said to be compatible if,

$$\lim_{n\to\infty} \varrho\left(fg_{x_n} ,\ gf_{x_n}\right) = 0 ,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} f_{x_n} = p = \lim_{n\to\infty} g_{x_n}$$

for some $p \in X$.

In 1984 Khan et al. introduced the concept of altering distance function and defined altering distance function as below.

Definition 2.3 : [25] A mapping

$$f : [\,0, \infty\,) \to [\,0, \infty\,)$$

is said to be an altering distance function, if it satisfies following conditions

(i) f is continuous and monotonically increasing

(ii) $f(x) = 0$ if and only if $x = 0$.

Fixed point results involving altering distances have been studied in [6, 11, 12].

Pant studied K-weakly commuting selfmaps in 1986.

Definition 2.4 : [29] Let (X, ϱ) be a metric space a pair (f, g) of selfmaps is said to be K-weakly commuting if there exists some $K > 0$ such that,

$$\varrho\,(fgx, gfx) \le K\varrho\,(\,fx, gx) ,$$

for all $x \in X$.

Jungck defined weakly commuting pair as follows.

Definition 2.5 : [22] Let (X, ϱ) be a metric space and f, g be self maps on X. Then we say that (f, g) is a weakly commuting pair (w.c.p.) or f, g are weakly commutative or f, g commute weakly if,

$$(fgx\,,\ gfx) \leq \varrho\,(fx\,,\ gx)\,,$$

for each $x \in X$.

Jungck and Rhoades defined weakly compatible selfmaps as below.

Definition 2.6 : [23] Let f and h be mappings from a metric space (X, ϱ) into itself. Then f and h are said to be weakly compatible if, they commute at their points of coincidence, that is, if $fa = ha$ for some $a \in X$, then $hfa = fha$. If we write $fa = ha = b$, then we say that b is a point of coincidence of (f, h).

In 2001 B. E. Rhoades defined the weakly contractive mapping as below.

Definition 2.7 : [31] Let (X, d) is a Metric Space, then a mapping $g : X \to X$ is said to be weakly contractive if for all $a, b \in X$,

$$d(\psi a,\ \psi b) \leq d(a, b) - \phi(d(a, b))\,,$$

where $\emptyset : [\,0, \infty) \to [0, \infty)$ is a continuous nondecreasing function satisfying $\phi(x) = 0$ if and only if $x = 0$. If $\phi(x) = (1 - p)x$, where $0 < p < 1$, then a weak contraction reduces to the Banach contraction.

3 COMMUTING SELFMAPS WITH UNIQUE FIXED POINT

We have obtained a new fixed point result for selfmappings defined on complete metric space satisfying contractive conditions that involves a function of two variables and acts on distances of two pair of points in a Metric spaces.

Ciric generalized the following theorem for unique fixed point.

Theorem 3.1 : (Ciric [8]) Let (X, d) be a complete metric space and f be a selfmap on X such that, for all $x, y \in X$

$$d\,(fx, fy) \leq \mu \max\{\,d\,(x, y), d\,(x, fx)\,,\ d\,(y, fy), d\,(x, fy), d\,(y, fx)\}$$

where $\mu \in (0,1)$ is some constant. Then f has a unique fixed point.

Jungck proved the following theorem for common fixed point.

Theorem 3.2 : (Jungck [21]) Let (X, d) be a complete metric space and f, g be commuting continuous selfmaps on X with $g(X) \subset f(X)$. Moreover, let there exist a constant $\mu \in (0,1)$ satisfying

$$d\,(gx, gy) \leq \mu\, d(\,fx, fy) \quad \text{for all } x, y \in X.$$

Then f and g have a unique common fixed point.

M. S. Khan proved the following theorem by using selfmap for unique fixed point.

Theorem 3.3 : (Khan [25]) Let (X, d) be a complete metric space and f be a self mapping on X, such that for all $x, y \in X$ and $0 \leq \alpha < 1$,

$$d(f_x, f_y) \leq \alpha\left(d(x, f_y)\, d(y, f_x)\right)^{1/2},$$

then f has a unique fixed point.

D. S. Jaggi proved the following theorem for unique fixed point.

Theorem 3.4 : (Jaggi [20]) Let f be a selfmap defined on a complete metric space (X, d), and is continuous at $t = 0$ if and only if $t = 0$. Again let f satisfies the condition

$$d(f(x), f(y)) \leq \frac{\lambda\, d(x, f(x))\, d(y, f(y))}{d(x, y)} + \mu\, d(\,x, y\,)$$

for all $x, y \in X$, $x \neq y$ and for some $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$, then f has a unique fixed point in X.

D. S. Jaggi further generalized theorem 3.4 for some integer m as follows.

Theorem 3.5 : (Jaggi [20]) Let f be a selfmap defined on a complete metric space (X, d), such that for some positive integer m, f satisfies the condition

$$d(f^m(x), f^m(y)) \leq \frac{\lambda\, d(x, f^m(x)).\, d(y, f^m(y))}{d(x,y)} + \mu\, d(x,y)$$

for all $x, y \in X$, $x \neq y$ and for some $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$. Further, if f^m is continuous then f has a unique fixed point.

K. Goebel and W. A. Kirk proved the following theorem for fixed point.

Theorem 3.6 : (K. Goebel and W. A. Kirk [18]) Let f be a selfmap on a complete metric space (X, d) and satisfy

$$[(\mu (f_x, f_y))^p + r(\mu(x, f_x))^q]^k + [(\mu (y, f_y))^p + r (\mu(y, f^2_x))^q]^k \leq \lambda [(\mu (x,y))^p + r(\mu (x, f_x))^q]^k + \lambda'[(\mu (y, f_y))^P + r (\mu (y, f_y))^q]^k$$

For all $x, y \in X$, where $p, k > 0$; $r, q \geq 0$ and $0 < \lambda < 1$; $0 < \lambda' \leq 1$, then f has a unique fixed point.

Theorem 3.7 : Let (X , d) be a complete metric space and let f be a continuous selfmap on X *and* g be a selfmap on X that commute with f. Also f *and* g satisfy conditions of definition 2.1 and theorem 3.2, then f *and* g have a unique common fixed point.

4 MAIN RESULT

We have generalized theorems 3.1 and 3.3 by using conditions in definitions 2.2 and 2.3 and obtained the results for unique fixed point as follows.

Theorem 4.1 : Let (X, d) be a compact metric space. Let f and g be commuting selfmaps of X such that fg is continuous. Again if

$$fx \neq gy \Rightarrow$$
$$d(fx, gy) < diam\, \{h(z) / z \in \{x, y\},$$

and $h \in C_{gf}$.

Then there is a unique point $x_0 \in X$ such that $x_0 = fx_0 = gx_0$.

Theorem 4.2 : Let f be a selfmap on a complete metric space (X, d) such that f^2 in continuous. And let $g: f(X) \to X$ be such that

$$gf(X) \subset f^2(X) \text{ and } g(f(x)) = f(g(x))$$

whenever both sides are defined .

Moreover, let there exist a number $\mu \in (0,1)$ such that for all $x, y \in f(X)$, conditions in definition 2.1 holds. Then f *and* g have a unique common fixed point .

Proof : At first we shall show that the theorem is true for $n = 2$. As f is a contraction, therefore we consider $\mu < 1$, then

$$d(f(x), f(y)) \leq \mu\, d(x, y)$$

We can apply f to $f(x)$ and $f(y)$ such that

$$d(f^2(x), f^2(y)) \leq \mu\, d(f(x), f(y)).$$

Since $d(f(x), f(y)) \leq \mu\, d(x, y)$

$$d(f^2(x), f^2(y)) \leq \mu\, d(f(x), f(y)) \leq \mu^2 d(x, y).$$

Thus, $d(f^2(x), f^2(y)) \leq \mu^2 d(x, y)$

Since $\mu < 1$, $\mu^2 < 1$ then f^n is a contraction.

Now as f^n is a contraction, therefore this implies that f^{n+1} is also a contraction.

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \mu^{n+1}\, d(f(x), f(y)) \leq \mu^{n+1} d(x, y).$$

So $d(f^{n+1}(x), f^{n+1}(y)) \leq \mu^{n+1} d(x, y)$.

Therefore by induction, the theorem is true for all n.

Again if $f(x) = x$, then

$$f^2(x) = f(f(x)) = f(x) = x\,.$$

Thus $f^n(x) = x$.

Therefore by induction, f^n is also a contraction. And hence f^n has a unique fixed point.

Theorem 4.3 : Let (X, d) be a complete metric space suppose f is contraction mapping and μ is a constant for f and μ^n is the constant for f^n. Then f^n is also a contraction and f^n has a fixed point.

Proof : At first we shall show that the theorem is true for $n = 2$. As f is a contraction, therefore we consider $\mu < 1$, then

$$d\big(f(x), f(y)\big) \leq \mu\, d(x, y)$$

Applying f to $f(x)$ and $f(y)$, we get

$$d\big(f^2(x), f^2(y)\big) \leq \mu\, d\big(f(x), f(y)\big).$$

Now as $d\big(f(x), f(y)\big) \leq \mu\, d(x, y)$,

therefore we get

$$d\big(f^2(x), f^2(y)\big) \leq \mu\, d\big(f(x), f(y)\big) \leq \mu^2 d(x, y).$$

This implies $d\big(f^2(x), f^2(y)\big) \leq \mu^2 d(x, y)$.

Again as $\mu < 1, \mu^2 < 1$, therefore f^n is a contraction.

Now as f^n is a contraction, therefore this implies that f^{n+1} is also a contraction.

$$d\big(f^{n+1}(x), f^{n+1}(y)\big) \leq \mu^{n+1}\, d\big(f(x), f(y)\big) \leq \mu^{n+1} d(x, y).$$

So $d\big(f^{n+1}(x), f^{n+1}(y)\big) \leq \mu^{n+1} d(x, y)$.

Therefore by induction, the theorem is true for all n. If $f(x) = x$, then

$$f^2(x) = f\big(f(x)\big) = f(x) = x\,.$$

Thus $f^n(x) = x\,.$

Therefore by induction, f^n is also a contraction. And hence f^n has a unique fixed point.

5 CONCLUSION

In the present paper we have used altering distance function, commuting selfmaps, weakly commuting pair, K-weakly commuting pairs for obtaining the unique common fixed point of selfmaps in complete metric spaces.

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