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Generalization of Banach, Goebel and Kirk Fixed Point Theorems

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Abstract:

The Banach contraction principle [1] is the first important result on fixed points for contractive type mappings. In 1969 Kannan [9], in 1972 Chatterjee [4] and in 2001 Rhoades B. E. [13] gave some interesting results on fixed points. Boyd and Wong [2] obtained more general fixed point theorem by replacing the decreasing function in the theorem of Rakotch [14] and it was further generalized by Nalawade and Dolhare U. P. [12]. In this paper our aim is to discuss about fixed point theory. We have also established fixed point theorem in complete Metric Space, which is a new generalized result in fixed point theory.

Keywords — **Fixed point, Fixed point theorem, Contraction mapping, Weakly contraction mapping.**

I. DEFINITIONS AND EXAMPLES

A point x which is mapped to itself under a map f, so that $f(x) = x$, such points are sometimes also called as invariant points or fixed points.

Definition 1: Let X be a non-empty set and f be self-map on X, then a point $x \in X$ is called a fixed point of f if $f(x) = x$.

Example 1: If f is selfmap on real numbers defined by $f(x) = x^2 - 3x + 4$, then 2 is a fixed point of f because $f(2) = 2$.

Example 2: $f(x) = x - 1$ has no fixed point as $f(x) \neq x$ for any x.

Definition 2: A Metric Space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$, we have

(M_1) d is real-valued, finite and non negative. (non-negativeness)

(M_2) $d(x,y) = 0$ if and only if $x = y$ (coincidence)

(M_3) $d(x,y) = d(y,x)$ (symmetry)

(M_4) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

Example 3: The set of all real numbers, taken with the usual metric defined by $d(x,y) = |x - y|$ is a Metric Space.

Example 4: If (X,d) is a Metric Space and $A \subseteq X$, then (A,d) is also a Metric Space.

Definition 3: Let (X,d) be a Metric Space, a sequence $\{x_n\} \in X$ is said to be Cauchy sequence if $d(x_m, x_n) \longrightarrow 0$ as $m,n \longrightarrow \infty$.

Definition 4: A Metric Space (X,d) is said to be a 'Complete Metric Space' if every Cauchy sequence in X converges to a point of X.

Example 5: The space $C[0,1]$ of all continuous real valued function on $[0,1]$ with metric ϱ defined by $\varrho(f,g) = \sup\{|f(x) - g(x)|;\ x \in [0,1]\}$ is Complete Metric Space.

Khan M. S. et al. defined the altering distance as follows:

Definition 5: (Khan M. S. et al. [11]) An altering distance is a mapping $f: [0,\infty) \longrightarrow [0,\infty)$ which satisfies,

(a) f is increasing and continuous, and

(b) $f(t) = 0$ if & only if $t = 0$.

Definition 6: A function $\emptyset : R \times R \longrightarrow R$ is said to satisfy condition (a) if

(i) $\emptyset$ is monotonic increasing in both the conditions

(ii) $\emptyset$ is continuous

(iii) $\emptyset(0,0) = 0$ and $\emptyset(\epsilon, 0) = 0$ implies $\epsilon = 0$.

Let $\emptyset(\epsilon,\epsilon)=0$. Then $\emptyset(\epsilon,0)\le\emptyset(\epsilon,\epsilon)=0$ or $\emptyset(\epsilon,0)=0$ which implies that $\epsilon=0$ by condition (iii).

Edelstein M. defined the contractive maps as follows

Definition 7: (Edelstein M. [7]) Let X be a Banach space. Then a selfmap F of X is called contractive if for every $x,y\in X$, with $x\neq y$, we have

$$d(F_x,F_y)<d(x,y).$$

To obtain a fixed point it is necessary to add the assumption that there exists a point $x\in X$ for which $\{F_x^n\}$ contains a convergent subsequence.

Rhoades B. E. defined **weakly contractive maps** as below

Definition 8: (Rhoades B. E. [13]) Let X be any Banach space, then a selfmap F of X satisfies the Banach contraction principle, if there exists a constant λ with $0\le\lambda<1$ such that for each $x,y\in X$, we have

$$\|F_x-F_y\|\le\lambda\|x-y\|. \qquad (1)$$

Mappings satisfying (1) possesses a unique fixed point, and the fixed point can be obtained by beginning at any point $x\in X$ and then by the repeated iteration of map F.

The inequality (1) can be written as

$$\|F_x-F_y\|\le\|x-y\|-p\,\|x-y\| \qquad (2)$$

where $p=a-\lambda$.

The inequality (2) can be extended naturally to the weakly contractive map as below.

Let X be a Banach space and S be a closed convex subset of X. Then a selfmap F of S is called weakly contractive if for every $x,y\in S$, we have

$$\|F_x-F_y\|\le\|x-y\|-\phi(\|x-y\|), \qquad (3)$$

where $\phi:[0,\infty)\to[0,\infty)$ is continuous and non-decreasing map satisfying $\phi(x)>0$, $\forall x\in[0,\infty)$, $\phi(0)=0$, and $\lim_{t\to\infty}\phi(t)=\infty$. If S is bounded, then the condition of infinity can be dropped.

II. Some Theorems

Theorem 1: Every continuous function from closed disk to itself has at least one fixed point.

Theorem 2: Every continuous function from closed ball of Euclidean space into itself has a fixed point.

Theorem 3: Every continuous function from convex compact subset X of a Euclidean space to X has a fixed point.

Theorem 4: Every continuous function from a Banach space X to X has a fixed point.

Rhoades B. E. proved the following fixed point theorem for weakly contractive maps.

Theorem 5: (Rhoades B. E. [13]) Let (X,d) be a complete Metric Space. And for some $\psi\in\Psi$, let F be a ψ-weak contraction on X, then the map F has a unique fixed point.

Singh S. P. defined the contractive mapping as follows

Definition 9: (Singh S. P. [15]) Let F be self-map on Metric Space X then F is said to be contraction mapping if

$$\varrho\,(F(x),F(y))\le\gamma\,\varrho\,(x,y),$$

for all $x,y\in X$ and for some real number γ such that $0<\gamma<1$.

According to a prominent theorem of Banach, if X is a Complete Metric Space and if F is a selfmap on X that satisfies

$$\varrho(Fx,Fy)\le\lambda\,\varrho(x,y) \qquad (4)$$

for some $\lambda<1$ and for all $x,y\in X$, then F has a unique fixed point, say x_0. Moreover the successive approximations $\{F_x^n\}$ converge to x_0 for $x\in X$.

However the condition

$$\varrho(F_x,F_y)<\varrho(x,y)$$

fails to ensure that F has a fixed point.

In the present paper we investigate some mappings that satisfy the condition

$$\varrho(F_x,F_y)\le\phi(\varrho(x,y)),$$

where ϕ is a mapping defined on the closure of the range of the function ϱ.

Banach proved the following theorem for fixed points.

Theorem 6: ([Banach] [1]) Let (X,d) be a complete Metric Space and let F be a contraction on X, then F has a unique fixed point.

Dutta P. N. and Choudhary B. S. introduced a new generalization of the contraction principle and proved the following theorem.

Theorem 7: (Dutta P. N. and Choudhury B. S. [6]) Let (X,d) be a complete Metric Space and let F be a selfmap on X which satisfies the condition

$$\phi\left(d(F_x,F_y)\right)\le\phi(d(x,y))-\varphi(d(x,y)), \qquad (5)$$

for some $\phi\in\Psi$ and $\varphi\in\Phi$ and for all x,y in X, then F has a unique fixed point.

Skof F. proved the following theorem for fixed point:

Theorem 8 : (Skof F. [16]) Let F be a selfmap on a complete Metric Space (X,d) and let $\phi \in \Phi$ satisfies the condition

$$\phi\left(d(F_x, F_y)\right) \le \alpha\, \phi(d(x,y)) + \beta\, \phi(d(x, F_x)) + \gamma\, \phi\left(d(y, F_y)\right),$$

for $0 \le \alpha + \beta + \gamma < 1$, and for all x, y in X, then F has a unique fixed point.

Kannan R. proved the following theorem for fixed point.

Theorem 9: (Kannan R. [10]) Let F be a selfmap on the complete Metric Space X i.e. $F: X \longrightarrow X$ is a mapping for which the condition

$$d\left(F_x, F_y\right) \le \lambda\{d(x, F_x) + d(y, F_y)\}, \qquad (6)$$

hold for all x, y in X, and $0 < \lambda < \frac{1}{2}$, then F has the unique fixed point in X.

Kannan R. proved the following theorem for fixed point.

Theorem 10: (Kannan R. [9]) Let (X,d) be a complete Metric Space and let $F: X \longrightarrow X$ satisfy the condition

$$d(Fx, Fy) \le r\{d(x, Fx) + d(y, Fy)\},$$

for all $x, y \in X$, where $r \in [0, \frac{1}{2})$, then F has a unique fixed point in X.

L.B. Ciric introduced the following notion of generalized contractions and proved the subsequent theorem.

Theorem 11: (Ciric [5]) Let F be a λ - generalized contraction of F-orbitally Complete Metric Space X into itself. Then

1) there is a unique point $u \in X$ which is a fixed point under F
2) $F_x^n \longrightarrow u$, for every $x \in X$
3) $d(F_x^n, u) \le \frac{\lambda^n}{1-\lambda} d(x, Fx)$

Caccioppoli generalized Banach contraction principle for constant C_n as follows.

Theorem 12: (Caccioppoli [3]) extends the Banach principle for mapping F converge in a Complete Metric Space X provided that for each $n \ge 1$, there exists a constant C_n such that

$$d\left(F^n(x), F^n(y)\right) \le C_n\, d(u,v)$$

for every $x, y \in X$, where $\sum C_n < \infty$ then F has a fixed point.

Rakotch , in 1962 , generalized Banach contraction principle for monotonically decreasing function as follows:

Theorem 13: (Rakotch [14]) Let X be a complete Metric Space and $f: X \longrightarrow X$ satisfies

$$f(f(x), f(y)) \le \lambda(d(x, y), d(x, y))$$

for all $x, y \in X$, where $\lambda: R^+ \longrightarrow [0,1]$ is monotonically decreasing, then f has a unique fixed point $\bar{x}$ and $\{f^n(x)\}$ converges to $\bar{x}$ for each $x \in X^n$.

Goebel K. and Kirk W. A. generalized following fixed point theorem.

Theorem 14: (Goebel K. and Kirk W. A. [8]) Let $f: X \longrightarrow X$ be a self-mapping from a complete Metric Space X to itself and satisfy

$$\left[(\delta(f_x, f_y))^p + r\left(\delta(x, f_x)\right)^q\right]^k + \left[(\delta(y, f_y))^p + r(\delta(y, {f_x}^2))^q\right]^k \le \lambda\left[(\delta(x,y))^p + r(\delta(x, f_x))^q\right]^k + \lambda'\left[(\delta(y, f_y))^p + r(\delta(y, f_y))^q\right]^k \qquad (7)$$

where $x, y \in X$; $p, k > 0$; $r, q \ge 0$ and $0 < \lambda < 1$, $0 < \lambda' \le 1$,

then f has a fixed point.

III. Main Results for Fixed Points

By taking small values of p, q, r and k, consider $p = q = r = k = 1$, we have generalized the theorem 14 as follows:

Theorem 15: Let $f: X \longrightarrow X$ be a self-mapping from a complete Metric Space X to itself and which satisfies following inequality

$$\emptyset(\delta(f_x, f_y), \delta(x, f_x)) + \emptyset(\delta(y, f_y), \delta(y, f_x^2)) \le \lambda\, \emptyset(\delta(x,y), \delta(x, f_x)) + \lambda'\, \emptyset(\delta(y, f_y), \delta(y, f_x)) \qquad (8)$$

where $x, y \in X$; $0 < \lambda < 1$, $0 < \lambda' \le 1$, then f has a unique fixed point.

Proof : Let us consider $x_0 \in X$ and consider a sequence $\{x_n\}$ defined by

$$x_n = f_{x_{n-1}} = f_{x_0}^n \qquad (9)$$

where $n = 1, 2, \dots$.

Putting $y = f_x$ in (8), we get

$$\emptyset(\delta(f_x, f_x^2), \delta(x, f_x)) + \emptyset(\delta(f_x, f_x^2), \delta(f_x, f_x^2)) \le \lambda\, \emptyset(\delta(x, f_x), \delta(x, f_x)) + \lambda'\, \emptyset(\delta(f_x, f_x^2), \delta(f_x, f_x)) \qquad (10)$$

Since $0 < \lambda' \le 1$ and $\emptyset$ satisfies the condition (b) in the definition 5, we have

$$\lambda' \emptyset(\delta(f_x, f_x^2), 0) \leq \lambda' \emptyset(\delta(f_x, f_x^2), \delta(f_x, f_x^2))$$
$$\leq \emptyset(\delta(f_x, f_x^2), \delta(f_x, f_x^2))$$

and from (10) we have

$$\emptyset\,(\delta(f_x, f_x^2), \delta(x, f_x)) \leq \lambda\, \emptyset\big(\delta(x, f_x), \delta(x, f_x)\big)$$
$$\leq \emptyset\big(\delta(x, f_x), \delta(x, f_x)\big) \quad (11)$$

which implies that

$$\delta(f_x, f_x^2) \leq \delta(x, f_x) \quad (12)$$

Putting $x = x_{n-1}$, we have

$0 \leq \delta(x_{n+1}, x_n) \leq \delta(x_n, x_{n-1}),\ \ n = 1, 2, \ldots$.

This shows that the sequence $\{\delta(x_n, x_{n+1})\}$ converges.

Let $\delta(x_n, x_{n+1}) = p$ (say) (13)

Then from (11) by putting $x = x_{n+1}$, we have

$$\emptyset\big(\delta(x_n, x_{n+1}), \delta(x_{n-1}, x_n)\big)$$
$$\leq \lambda\, \emptyset(\delta(x_{n-1}, x_n), \delta(x_{n-1}, x_n)) \quad (14)$$

as $n \longrightarrow \infty$ *and* $\emptyset$ is continuous, then $\emptyset(p, p) = 0$ which implies that $p = 0$.

By (9), $\lim\limits_{x \to \infty} \delta(x_n, x_{n+1}) = p$ (15)

If there exist $\epsilon > 0$ and consider the subsequences $\{x_{m(p)}\}$ *and* $\{x_{n(p)}\}$ of the sequence $\{x_n\}$ such that $m(p) < n(p)$.

$$\therefore\ \ \delta(x_{n(p)}, x_{m(p)}) \leq \epsilon \quad (16)$$

As $p \longrightarrow \infty$ and using (15) and (16), we obtain

$$\lim_{p \to \infty} \delta\big(x_{n(p)}, x_{m(p)-1}\big) = \epsilon\,.$$

As $p \longrightarrow \infty$ and consider (15), $x_n \longrightarrow x$ and by continuity of $\emptyset$ then we have

$$\emptyset(\delta(x, f_x), 0) + \emptyset(\delta(x, f_x), 0)$$
$$\leq \lambda\, \emptyset(0, 0) + \lambda\, \emptyset\,(\delta(x, f_x), 0).$$

Which implies that

$\emptyset(\delta(x, f_x), 0) \leq\ \lambda\, \emptyset(0, 0)$ in this case $0 < \lambda' < 1$

$$\leq\ \lambda\, \emptyset(\delta(x, f_x), 0)$$

using condition (ii).

Similarly $\emptyset(\delta(x, f_x), 0) = 0$, $0 < \lambda < 1$.

So that $\delta(x, f_x) = 0$ by condition (ii) in definition 6.

Similarly $\emptyset(\delta(x, f_x), 0) = 0$, $0 < \lambda < 1$.

So that $\delta(x, f_x) = 0$ by condition (iii) in definition 6.

Which implies that $f_x = x$.

This shows that f has a unique fixed point.

We also have generalized the Banach theorem 6 for weakly contractive maps as follows:

Theorem 16: Let (X, d) be a complete Metric Space, and F be the weakly contractive map. Then F has a unique fixed point t in X.

Proof: Firstly we shall establish the existence of the fixed point. Let us consider $x_0 \in X$ and we define $Fx_n = x_{n+1}$.

Then from (3), we have

$$d(x_{n+1}, x_{n+2}) = d(Fx_n, Fx_{n+1})$$
$$\leq d(x_n, x_{n+1}) - \emptyset\big(d(x_n, x_{n+1})\big),$$

where $\lambda_{n+1} \leq \lambda_n - \emptyset(\lambda_n) \leq \lambda_n$ (17)

Therefore $\{\lambda_n\}$ is a non-negative non-increasing sequence and hence it possesses a limit $\lambda \geq 0$. Now we have to show that F is a selfmap of the ball $B(x_n, \epsilon)$. Let $x \in B(x_n, \epsilon)$, then we have two cases

Case I: Let $d(x, x_n) \leq \frac{\epsilon}{2}$,

$$d(Fx, x_n) \leq d(Fx, Fx_n) + d(Fx_n, x_n)$$
$$\leq d(x, x_n) - \phi\,(d(x, x_n) + d(x_{n+1}, x_n))$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon\,.$$

Case II: Let $\frac{\epsilon}{2} < d(x, x_n) \leq\ \epsilon$, then

$$\phi\big(d(x, x_n)\big) \geq \phi\left(\frac{\epsilon}{2}\right),$$

therefore

$$d(Fx, x_n) \leq d(x, x_n) - \psi\, d(x, x_n) + d(x_{n+1}, x_n)$$
$$\leq d(x, x_n) - \psi\left(\frac{\epsilon}{2}\right) + \psi\left(\frac{\epsilon}{2}\right)$$
$$= d(x, x_n)$$
$$\leq\ \epsilon\,.$$

Since F is a selfmap of $B(x_n, \epsilon)$, it follows that each $x_n \in B(x_n, \epsilon)$ for $n > 0$.

Now as ϵ is arbitrary, therefore $\{x_n\}$ is cauchy sequence, and as X is complete, therefore $\{x_n\}$ is convergent. Again, the continuity of F implies that the limit is a fixed point. Finally the uniqueness of the fixed point follows from (3). Hence the proof.

IV. CONCLUSION

In the present paper we have used Banach principle, selfmaps, contraction mapping to find out fixed point of selfmaps in Complete Metric Space.

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